Uncertainty Relations for Two Dimensional Quantized Electromagnetic Potential

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Abstract

The canonical quantization of flux is performed. It is shown that according to the canonical flux quantization there must be a new uncertainty relation: $e\Delta A_m.\Delta x_m \geq \hbar$ where A_m and $\Delta x_m \geq l_B$ are the electromagnetic gauge potential, the position uncertainty and the magnetic length, respectively. Other arguments in favour of this uncertainty relation are also discussed.

The flux quantization is described according to the relation: $\oint e A_m dx^m = \int \int e F_{mn} dx^m \wedge dx^n = \Phi = Nh, N \in \mathbf{Z}$ where A_m and F_{mn} are the electromagnetic potential and the magnetic field strength and $m, n = 1, 2$. We prove that in this case there must be a new uncertainty relation $e\Delta A_m . \Delta x_m \geq \hbar$ which is equivalent to the canonical flux quantization according to the quantum commutator postulate $e[\hat{A}_m, \hat{x}_m] = -i\hbar$. Hereby x_m can be considered either as the coordinates of the centre of cyclotron motion or as the relative coordinates around the centre [1].

To begin we show that such a commutator can be considered as a result of electronic behaviour in magnetic fields:

From the usual requirement in flux quantization that the electronic current density $j_m = ne\hat{V}_m = \Psi^*(\hat{p}_m - e\hat{A}_m)\Psi$ must vanish in the region where the contour integral $\oint A_m dx^m$ takes place [2], one concludes that *in this region* $[\hat{V}_m , \hat{x}_m] = 0$. This implies that in this region $[\hat{p}_m , \hat{x}_m] = e[\hat{A}_m , \hat{x}_m]$ or that $e[\hat{A}_m s, \hat{x}_m] = -i\hbar$.

Moreover, it is also known that for the cyclotron motion of electrons, the coordinate operators of relative coordinates are non-commuting. Thus, one has $[\hat{x}_m , \hat{x}_n] = -il_B^2 \epsilon_{mn}$ for the relative cyclotron coordinates, where l_B is the magnetic length [1]. This is an interesting example of the non-commutative geometry of configuration space in quantum theory. Now, the mentioned commutator $[\hat{A}_m , \hat{x}_m]$ is proportional to this commutator in the usual Landau gauge $A_m = Bx^n \epsilon_{mn} , \epsilon_{mn} = -\epsilon_{nm} = 1$ which was introduced to study the behaviour of electrons in magnetic fields [1]. Therefore, in view of this proportionality one has indeed $[\hat{A}_m , \hat{x}_m] = B\epsilon_{mn}[\hat{x}_n , \hat{x}_m] = -il_B^2 \cdot B = -i\frac{\hbar}{e}$ for $C = 1$.

It should be mentioned also that the usual argument, that the electromagnetic potential A_m is a function of x^m and therefore the operators $\hat{A}_m$ and $\hat{x}_m$ must commute with each other, does not apply to the case of flux quantization:

Here A_m is not a function of x^m, but it is given either by $A_m = B \cdot x^n \epsilon_{mn}$ *within* the flux surface where $\epsilon_{mn} F_{mn}(A_m)_{(surface)} = B$ is constant, or it is given by an electromagnetic pure gauge potential $\tilde{A}_m := \partial_m \phi$ in the contour region [2]. In these both related cases which are relevant for the flux quantization the electromagnetic potential A_m is not a function of x_m. Therefore, there is no a periori reason for the commutativity of the operators $\hat{A}_m$ and $\hat{x}_m$.

After these consistency arguments for the non-triviality of mentioned commutator and uncertainty rela-

tions we give a more rigorous prove for their existence according to the canonical quantization structure of the flux action functional.

We will show that, indeed for the true phase space variables of the two dimensional electromagnetic system of flux quantization, the commutator of related operators is non-trivial and so there exist equivalent uncertainty relations. The key point is the choise of correct phase space, i. e. the choise of true canonical conjugate variables for the electromagnetic system under consideration, which has to be quantized in order to describe the flux quantization.

The point of departure is the flux quantization relation for electromagnetic system:

$$\int \int e F_{mn} dx^m \wedge dx^n = \Phi = Nh \tag{1}$$

This quantization should be, in principle, describable by the canonical quantization of the classical action functional $S_{(Cl)}^{(flux)}$ which is given naturally by the flux quantization relation (1):

$$S_{(Cl)}^{(flux)} = \int \int e F_{mn} dx^m \wedge dx^n = \int \int e dA_n \wedge dx^n , \tag{2}$$

where we used $dA_n := \partial_m A_n dx^m$.

To quantize the phase space of a classical system which is represented by an action functional $S_{(Cl)}$, one should determine first the canonical conjugate variables of phase space and then one should postulate the quantum commutator for operators which are related to these variables. Now to determine the phase space space variables of the system which is represented by the action functional $S_{(Cl)}$ one should compare it with the canonical action functional:

$$S_{(Cl)}^{(canon)} = \int \int dp_m \wedge dx^m , \tag{3}$$

of the same dimension [3].

The comparison between $S_{(Cl)}^{(flux)}$ in (2) and $S_{(canon)}$ in (3) shows that the phase space of our system which is represented by $S_{(Cl)}^{(flux)}$ has the set of canonical conjugate variables: $\{eA_m, x^m\}$.

Then, the globally Hamiltonian vector fields of our system which has the symplectic 2-form $\omega = edA_n \wedge dx^n = eF_{mn}dx^m \wedge dx^n$ are given by the following differential operators [4], [5]:

$$X_{A_m} = \frac{\partial}{\partial x^m} \quad , \quad X_{x^m} = -\frac{\partial}{\partial A_m} \tag{4}$$

Moreover, the quantum differential operators on the quantized phase space of this system should be proportional to these vector fields by a complex factor, i. e. usually by $(-i\hbar)$, and so they should be given by $\hat{A} = -i\hbar\frac{\partial}{\partial x^m}$ and $\hat{x} = i\hbar\frac{\partial}{\partial A_m}$.

On the other hand, the actual quantized phase space of a quantum system should be polarized in the sense that the wave function of the system should be a function of only half of the variables of the original phase space [4]. This means that in general Ψ is either in the $\Psi(p_i, t)$- or in the $\Psi(x^i, t)$ representation. Then, the half of quantum operators which are related to the variables in Ψ act on Ψ just by the multiplication with these variables and the second half of quantum operators act on it by the action of quantum differential operators discussed above. In other words, as it is well known, for example in the $\Psi(p_i, t)$ representation the acting operators are given by $\hat{x}^i = -i\hbar X_{x^i}^{(canon)} = i\hbar\frac{\partial}{\partial p_i}$ and $\hat{p}_i = p_i$, which have the correct commutators: $[\hat{p}_i \ , \ \hat{x}^j] = -i\hbar\delta_i^j$. The same is true also in the $\Psi(x^i, t)$ representation for the $\hat{x} = x$ and $\hat{p}_i = -i\hbar\frac{\partial}{\partial x^i}$ operators.

In our case where in view of the neccessary polarization the wave function of $\{A_m, x^m\}$ system is either in $\Psi(A_m, t)$ or in $\Psi(x^m, t)$ representation, the quantum operators are given either by the set $\{\hat{A}_m = A_m \ , \ \hat{x}_m = -i\hbar X_{x^i} = i\hbar\frac{\partial}{\partial A_m}\}$ or by the set $\{\hat{A}_m = -i\hbar X_{A_m} = -i\hbar\frac{\partial}{\partial x^m} \ , \ \hat{x}^m = x^m\}$, respectively.

In both representations the commutator between the quatum operators is given by $(-i\hbar)$:

$$e[\hat{A}_m \ , \ \hat{x}_n]\Psi = -i\hbar\delta_{mn}\Psi \tag{5}$$

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Equivalently, we have according to quantum mechanics a true uncertainty relation for A_m and x_m, i. e.:

$e\Delta A_m \cdot \Delta x_m \geq \hbar$.

In other words, to describe the flux quantization according to the canonical quantization sheme, one has to consider the commutators (5) and also the equivalent uncertainty relations $e\Delta A_m \cdot \Delta x_m \geq \hbar$.

In the Landau gauge there should be also an equivalent uncertainty relation which is given by:

$eB\Delta x_m.\Delta x_n \geq \hbar|\epsilon_{mn}|$, i. e. for $m \neq n$. This uncertainty relation is related to the $eB[\hat{x}_m , \hat{x}_n] = -i\hbar\epsilon_{mn}$ commutator. The same uncertainty relation can be obtained also from the original uncertainty relations, if we use the Landau gauge $\Delta A_m = B.\Delta x^n \epsilon_{mn}$.

Moreover, the electromagnetic gauge potential have according to the uncertainty relations $e\Delta A_m \cdot \Delta x_m \geq \hbar$ a maximal uncertainty of $(\Delta A_m)_{max} = \dfrac{\hbar}{el_B}$ for the case $\Delta x_m = l_B$. Hence, using the Landau gauge one obtains the definition of magnetic length $l_B^2 = \dfrac{\hbar}{eB}$ which proves the consistency of this approach.

Footnotes and references

References

[1] Landau-Lifschitz, "Quantum Mechanics (Non-Relativistic Theory)" Vol.III (Pergamon Press 1987).

H. Aoki: Rep. Prog. Phys. 50, (1987), 655.

[2] In the usual derivation of flux quantization the integration path is chosen in a region where the electronic current density is zero. The vanishing of current density in a region, i. e. here in the contour region where the integral $\oint A_m dx^m$ is performed, is equivalent to the definition of the electromagnetic potential as a pure gauge potential in this region, i. e. $A_{m(contour)} = \tilde{A}_m$,

$F_{mn}(\tilde{A}_m)_{(contour)} = 0$.

[3] This procedure of canonical comparison to identify the phase space variables of a new system for its quantization is the very general and usual one which is used for any canonical quantization, e. g. for the quantization of the (3+1) decomposed Yang-Mills action functional, Maxwell's electrodynamics, the (2 + 1) decomposed Chern-Simons action functional, etc.

[4] N. Woodhouse,"Geometric Quantization", (Clarendon Press, 1980, 1990) Oxford University.

[5] According to the geometric quantization [4] the classical Hamiltonian vector fields related to the canonical conjugate variables of our $\{A_m, x^m\}$ phase space are given in general by:

$$X_{A_m} = \frac{\partial A_m}{\partial A_n} \frac{\partial}{\partial x^n} - \frac{\partial A_m}{\partial x^n} \frac{\partial}{\partial A_n} \qquad , \qquad X_{x^m} = \frac{\partial x^m}{\partial A_n} \frac{\partial}{\partial x^n} - \frac{\partial x^m}{\partial x^n} \frac{\partial}{\partial A_n}$$

Furthermore, the inner product of any globally hamiltonian vectorfield X_f with the symplectic 2-form of the system ω, should result in: $< X_f, \omega >= -df$.